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Technical Note

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Determination of Unsteady Heat Release Distribution in Unstable Combustor from Acoustic Pressure Measurements

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Introduction

DETRIMENTAL combustion instabilities in propulsion or power generation systems often occur when oscillations in heat release excite one or more of the natural acoustic modes of the combustion chamber.[1] Determining the characteristics of combustion driving processes in these unstable combustors remains one of the most important and unresolved problems in the field of combustion instability. Developing such an understanding often requires characterizing a number of processes occurring in the combustion zone, including chemical kinetics, fluid mechanics, and transport processes. Clearly, measuring the flowfield variables, e.g., velocity, temperatures, and species concentrations, that will characterize these processes in the combustion zone of an unstable combustor would tax the capabilities of most laboratories and would be impossible in practical combustors where optical windows and access for the required number of probes are not available. Consequently, approaches for determining the characteristics of the unsteady combustion process that do not require extensive measurements are needed.

Unsteady heat addition in a combustor affects the characteristics of the combustor's acoustic pressure, velocity, temperature, and density fields. Because it is often much easier to measure the characteristics of the excited acoustic pressure field, this Note explores the possibility of using acoustic pressure measurements to determine information about the unknown combustion process heat source. Specifically, the feasibility of recovering the spatial distribution of the amplitude and phase of the unsteady combustion process heat release in an unstable combustor from measurements of the spatial dependence of the acoustic pressure is investigated.

Such a technique was first attempted by Ramachandra and Strahle[2] and Ramachandra[3] to determine the distribution of the fluctuating heat release rate in an open premixed flame. They measured the spatial distribution of the acoustic pressure and then used linear acoustic theory to determine the spatial distribution of the fluctuating heat release. The accuracy of this calculated heat release distribution was assessed by comparison with radicals radiation measurements. Although encountering difficulties due to the sensitivity of the solution to measurement errors, they[2,3] reported that the technique was successfully used to recover the heat release distribution in the investigated flame.

Subsequently, Chao and Strahle[4] and Chao[5] attempted to extend the technique to recover the heat release distribution in a gas turbine combustor. They found that they were able to recover the heat release distribution only in certain frequency bands and concluded that the method was generally not feasible. In his thesis, Chao[5] commented that "the reasons why such poor results were obtained… are still not very clear." Unfortunately, without a clear understanding of his results, it is difficult to ascertain whether his conclusions are apparatus specific or reflect general limitations of the technique. Thus, the applicability of his results to other applications and combustion configurations is unclear.

This Note reconsiders this problem to better understand the previous results and to arrive at some general conclusions regarding the feasibility of the technique. The "Theoretical Background" section begins from a general formulation of the conservation equations and demonstrates that it is theoretically possible to recover the heat release from pressure measurements. In the Analysis and Discussion sections, issues of sensitivity of the solution to measurement errors are considered and the physical processes responsible for the prior experimental results are explained. The Note closes by drawing conclusions about the feasibility of using the technique in other unstable combustors such as those used in rocket motors or afterburners.

Theoretical Background

This section describes a theoretical analysis showing that, given certain assumptions to be outlined, it is theoretically possible to determine the spatial distribution of the unsteady heat release from acoustic pressure measurements. The investigated combustor is assumed to consist of a long and narrow duct of constant cross section and is shown in Fig. 1. A description of the unsteady flowfield in this combustor can be obtained from the mass, momentum, and energy conservation equations.[6] By the assumption that there is a one-dimensional, inviscid, perfect gas flow and that only a combustion process heat addition source is present in the combustor, the conservation and state equations may be written in the following form:

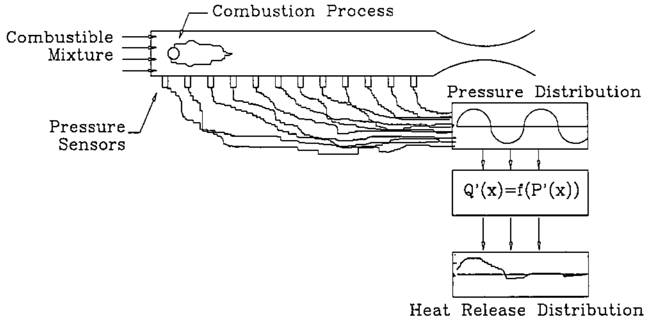

Fig. 1 Schematic of a proposed experimental setup for determining the unsteady combustion process heat addition from acoustic pressure measurements.

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$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \tag{1}$$

$$\rho\frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} \tag{2}$$

$$\frac{\partial[\rho(e + u^2/2)]}{\partial t} + \frac{\partial[\rho u(e + u^2/2)]}{\partial x} + \frac{\partial(\rho u)}{\partial x} = Q \tag{3}$$

$$p = \rho R T \tag{4}$$

where ρ, u, e, x, t, and Q are the density, axial velocity, specific energy, x location, time, and heat addition, respectively. To simplify the analysis, it will be assumed that the amplitude of the oscillations is small, i.e., $p'/\bar{p} \ll 1$, where barred and primed quantities describe mean and fluctuating variables, respectively; that the mean flow Mach number is small, i.e., $\bar{M}^2 \ll 1$, implying that $\bar{p} \approx$ const; and that the solutions have a harmonic time dependence, given by $\exp(i\omega t)$. Using these assumptions, the linearized versions of Eqs. (2) and (3) can be put in the following form:

$$i\omega\bar{\rho}u' + \bar{\rho}\bar{u}\frac{\mathrm{d}u'}{\mathrm{d}x} + \frac{\mathrm{d}p'}{\mathrm{d}x} + \bar{\rho}u'\frac{\mathrm{d}\bar{u}}{\mathrm{d}x} = 0 \tag{5}$$

$$\frac{(i\omega + \gamma\,\mathrm{d}\bar{u})}{\mathrm{d}x}p' + \bar{u}\frac{\mathrm{d}p'}{\mathrm{d}x} + \gamma\bar{p}\frac{\mathrm{d}u'}{\mathrm{d}x} = Q'(\gamma - 1) \tag{6}$$

The momentum equation (5) can be integrated to yield the following solution for $u'(x)$ in terms of $p'(x)$:

$$u' = \frac{1}{Z}\left(\left(\frac{1}{M}p'_0 + Zu'_0\right)\exp\left[-\int_0^x f(x')\,\mathrm{d}x'\right] - \frac{1}{M}p' + \left\{\exp\left[-\int_0^x f(x')\,\mathrm{d}x'\right]\Big/M\right\} \times \int_0^x p'f(x')\exp\left[\int_0^{x'} f(x'')\,\mathrm{d}x''\right]\mathrm{d}x'\right) \tag{7}$$

where $Z = \bar{\rho}\bar{c}$, $f(x) = (ik/M) + \mathrm{d}(\ell n\,\bar{u})/\mathrm{d}x$, where c is the speed of sound, M is the Mach number, and $k = \omega/c$. Substituting this expression and its derivative into Eq. (6) and then solving the resulting expression for $Q'(x)$ yield the following result:

$$Q'(x) = \frac{1}{\gamma - 1}\left(\left[\frac{ikc(1 + M^2)}{M^2} + \frac{\mathrm{d}\bar{u}}{\mathrm{d}x}\frac{(\gamma M^2 + 1)}{M^2}\right]p' - \frac{c}{M}(1 - M^2)\frac{\mathrm{d}p'}{\mathrm{d}x} + f(x)\exp\left[-\int_0^x f(x')\,\mathrm{d}x'\right] \times \left\{\frac{c}{M}\int_0^x p'f(x')\exp\left[\int_0^{x'} f(x'')\,\mathrm{d}x''\right]\mathrm{d}x' - c\left(\frac{1}{M}p'_0 + Zu'_0\right)\right\}\right) \tag{8}$$

Equation (8) shows that it is possible, in principle, to determine the unknown $Q'(x)$ provided that the mean combustor properties, boundary conditions, and $p'(x)$ are known. However, the terms in Eq. (8) do not lend themselves to simple physical interpretation. An examination of Eq. (8) shows that $Q'(x)$ depends on the boundary conditions, the acoustic pressure, the first derivative of the acoustic pressure, and a term containing the integral of the acoustic pressure. Each of these terms are weighted by coefficients dependent on the mean properties of the combustor.

Analysis

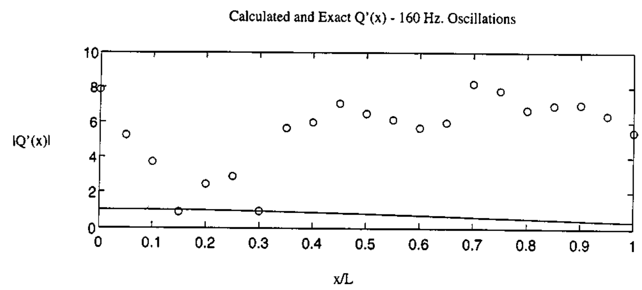
Whereas the preceding analysis demonstrated that it is theoretically possible to explicitly determine $Q'(x)$ from knowledge of the acoustic pressure and mean flow quantities, it provided little information about the properties of the solution and its sensitivity to errors in the measured $p'(x)$ distribution. Without some understanding of these issues, there is no assurance that a straightforward use of measured pressure data in Eq. (8) will yield a meaningful result.

To examine these issues, some computational solutions of the described equations were obtained. These solutions were obtained by carrying out the following procedure. First, the quantities $Q'(x)$, the mean flow quantities, and boundary conditions were specified and used to numerically solve Eqs. (5) and (6) for the resultant pressure distribution $p'(x)$. The fluctuating heat release was both assumed to have some explicit distribution, $Q'(x)$, i.e., a driven problem, or was assumed to be a function of the local pressure and velocity, i.e., $Q'(x) = Rp'(x) + Su'(x)$, i.e., an eigenvalue problem, where R and S are complex, constant response functions that determine the amplitude and phase relationships between the heat addition and pressure and velocity oscillations, respectively.

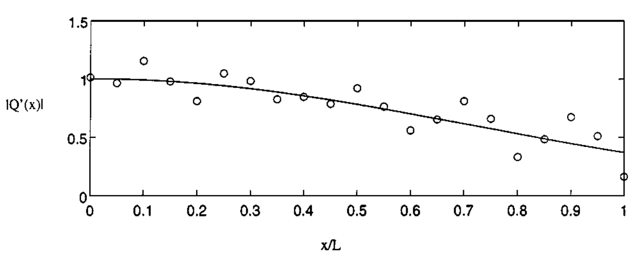
The results to be discussed were obtained for the case where the heat release distribution $Q'(x)$ was specified explicitly. Extensive results for the eigenvalue problem were also obtained but are not included here for lack of space. However, the results from the driven problem are sufficient to understand the important features of the problem.

Next, it was assumed that the calculated $p'(x)$ represented a pressure distribution measured in an experimental setup. Small errors were added to the calculated $p'(x)$ to evaluate the effect of measurement errors on the determination of $Q'(x)$. The modified, i.e., noisy, pressure distribution was then substituted into Eq. (8), and the resulting expression was solved for $Q'(x)$.

As expected, when perfect measurements of $p'(x)$ were assumed, Eq. (8) yielded the exact distribution of $Q'(x)$ that had been substituted into Eqs. (5) and (6) to determine the distribution of $p'(x)$. However, when errors were added to $p'(x)$, the situation dramatically changed. The calculated source distribution could be very sensitive to errors. For example, the addition of errors to $p'(x)$ with amplitudes as small as 0.01% of the amplitude of $p'(x)$ resulted in meaningless, wildly varying solutions for $Q'(x)$ in some cases. This sensitivity strongly depended on the frequency of oscillations in the combustor. For example, $Q'(x)$ could be determined quite accurately when errors were added to $p'(x)$ in situations where the frequency of oscillation was sufficiently removed from one of the combustors natural frequencies. However, if the combustor was oscillating near one of its resonant frequencies, the added error produced meaningless results. Figure 2 presents the results from some



a) Oscillations are near fundamental acoustic mode of the combustor



b) Oscillations are halfway between the fundamental and first harmonic acoustic modes of the combustor

Fig. 2 Comparison of the exact (line) and recovered (circles) $Q'(x)$ at two different driving frequencies; natural acoustic frequencies of the combustor were f = 150, 450, . . . , Hz.

of these calculations. Figure 2 shows that the frequency of oscillation significantly affects the accuracy of the solution. In the first case, the oscillations are occurring very near a resonant frequency of the combustor, and it can be seen that the recovery of the heat release is quite poor. In the other case, the oscillations are occurring halfway between the fundamental and first harmonic frequencies of the combustor, and the recovery of the heat release is very good.

Other computations, not reported here, showed similar results, regardless of the assumed boundary conditions or the mean combustor parameters, e.g., mean temperature distribution, flow velocity. There is one exception where the boundary conditions significantly influence the solution sensitivity. If the boundaries are acoustically nonreflecting, i.e., $p'/u'|_{x=0,L} = \bar{\rho}\bar{c}$, all disturbances are transmitted out of the region. This problem is equivalent to one where the heat release occurs in free space.

Discussion

This section further considers the discussed results and the previous experimental investigations[2–5] and then describes the implications of these findings on the feasibility of the experimental technique that is the focus of this Note. For the subsequent analysis, note that both the heat release and the acoustic properties of the duct, i.e., the boundary conditions, affect the measured pressure. Although the pressure distribution is affected by the heat release and the acoustic boundary conditions, the assumed linearity of the problem makes their relative contribution straightforward to account for. That is, the measured pressure consists of a linear superposition of the homogeneous solution and the particular solution of the wave equation [accounting for the effects of the boundary conditions and $Q'(x)$, respectively]. Consequently, the solution for the pressure is given by

$$p'(x) = p'(x)_{\text{hom}} + p'(x)_{\text{part}} \tag{9}$$

In the specific circumstance of a duct with no mean flow and a homogeneous medium, the solution for $p'(x)$ can be obtained from the following integral formulation[7] explicitly showing the form of each of the terms in Eq. (9):

$$p'(x) = \left[G(x, x_0)\frac{\mathrm{d}p'(x)}{\mathrm{d}x} - p'(x)\frac{\mathrm{d}G(x, x_0)}{\mathrm{d}x} \right]_0^L + \int_0^L G(x, x_0)\tilde{Q}'(x_0)\,\mathrm{d}x_0 \tag{10}$$

where $\tilde{Q}'(x_0) = [ik(\gamma - 1)Q'(x_0)]/c$ and $G(x, x_0)$ is the Green's function that equals the pressure at a location x due to a point source at a location x_0. Note that the first and second terms on the right-hand side of Eq. (10) describe the contribution of the boundary conditions and the heat source, respectively.

If the unsteady heat release occurs in free space, all excited acoustic disturbances will simply radiate to infinity. The pressure distribution will then take the following form:

$$p'(x) = p'(x)_{\text{part}} = \int_V G(x, x_0)\tilde{Q}'(x_0)\,\mathrm{d}x_0 \tag{11}$$

Note that this was the problem considered by Ramachandra and Strahle[2] and Ramachandra[3] in their investigation. This problem is similar to a number of other inverse problems studied in engineering, such as the inverse heat conduction problem.[8] These problems are considered to be ill posed because it can be shown[9] that an arbitrarily small change in $p'(x)$ can produce an arbitrarily large different distribution of $Q'(x)$. This characteristic of ill-posed problems is referred to as lack of stability[9] and reflects the sensitivity of the solution to measurement errors.

The physics behind the sensitivity of the solution to Eq. (11) can be understood by considering the effect of a flame sheet on the acoustic pressure. The expansion of reacting gases as they traverse the flame causes a jump in the acoustic velocity and, thus, a jump in the derivative of the acoustic pressure (see Fig. 3). Expressions for the jumps of these quantities are given by[1]

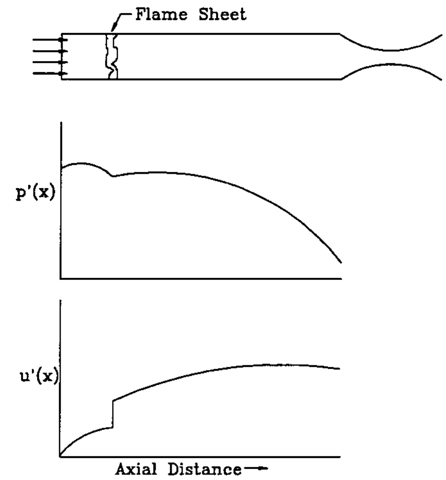


Fig. 3 Schematic of a flame sheet in a combustor, its acoustic pressure, and velocity fields.

$$\Delta u' = \frac{\gamma - 1}{\gamma \bar{p}} Q', \qquad \Delta \frac{\mathrm{d}p'}{\mathrm{d}x} = i\omega\rho\frac{\gamma - 1}{\gamma \bar{p}} Q' \tag{12}$$

Assume that the magnitude of the heat release in this flame sheet must be determined from pressure measurements. This will require calculating the derivative of the pressure just to the left and right of the flame and relating the difference to the magnitude of the heat release. It is well known that the numerical differentiation required in this procedure can significantly amplify small errors.[9] Note, however, that Eq. (12) suggests that the unsteady heat release can be deduced much more readily from acoustic velocity measurements. Such measurements have been discussed by Micci.[10]

The preceding analysis showed that solving for $Q'(x)$ in free space from Eq. (11) is difficult because the problem is ill posed. The sensitivity of the solution to measurement errors is further increased by the effects of the boundaries. This can be seen by rearranging Eq. (9):

$$p'(x)_{\text{part}} = p'(x) - p'(x)_{\text{hom}} \tag{13}$$

or using the formulation in Eq. (10):

$$\int_0^L G(x, x_0)\tilde{Q}'(x_0)\,\mathrm{d}x_0 = p'(x) - \left[G(x, x_0)\frac{\mathrm{d}p'(x)}{\mathrm{d}x} - p'(x)\frac{\mathrm{d}G(x, x_0)}{\mathrm{d}x} \right]_0^L \tag{14}$$

Equations (13) and (14) show that the heat release distribution is related to the difference between the total pressure and the homogenous solution for the pressure (which accounts for the effects of the boundaries). To understand why this enhances the solution sensitivity to errors, consider a hypothetical experiment in which an unsteady heat source with a gradually increasing amplitude is placed in a combustor. If no heat source is present in the combustor, the total pressure will simply equal the homogeneous solution of the wave equation. The resultant solution describes oscillations of one or more natural acoustic modes of the duct. If an unsteady heat source of very small magnitude is placed in the combustor, the total and homogeneous solutions will not be equal, although their difference will be small. This difference is related to the amplitude and phase distribution of the unsteady heat release. Consequently, if $p'(x)$ and $p'(x)_{\text{hom}}$ cannot be exactly determined, small errors in their measurement will produce very large errors in their difference, thus rendering the source calculation highly sensitive to these errors. (This sensitivity is best illustrated by an example. Consider a quantity Z that is determined by the difference between two numbers, i.e., $Z = X - Y$, where $X = 1000$, $Y = 998$, and, thus, $Z = 2$. Suppose that X has a 1% error so that $X_{\text{measured}} = 1010$, and, thus, $Z_{\text{measured}} = 12$. Thus, a 1% error in measurement of X gives a 500% error in determining Z.) If the magnitude of the source increases, the difference between the total and homogenous solutions for the pressure will increase, which reduces the effect of measurement errors on the calculated $Q'(x)$.

This discussion implies that the solution to the investigated problem is least sensitive to errors when there is no contribution to $p'(x)$ from the boundary's, that is, when the unsteady heat release does not occur in a confined region but in a free space, i.e., the problem considered by Ramachandra and Strahle[2] and Ramachandra[3] with a solution described by Eq. (11). Practically, to approach this situation, the contributions of the homogeneous and particular solutions to the total $p'(x)$ should be as small and large as possible, respectively. This conclusion indicates that to reduce the sensitivity of the solution to errors, the combustor should be oscillating at frequencies as far as possible from its natural frequencies because at these frequencies the homogeneous solution of a driven problem can become very large. In other words, sensitivity of the solution to measurement errors is infinitely large for an undamped combustor driven at one of its natural frequencies, decreases as the frequency approaches off resonance, and is minimum for a combustion process occurring in a free field.

Conclusions

This discussion is supported by the previous investigations described in the Introduction. The presented analysis shows that the sensitivity of the solution is smallest if the heat release occurs in a free field. As noted, this was the situation considered by Ramachandra and Strahle[2] and Ramachandra,[3] who were able to recover the heat release distribution of an open flame with reasonable success. For a bounded region, it was shown that minimum and maximum sensitivity of the solution occurred at off-resonant and resonant frequencies of oscillation, respectively. The experimental results of Chao and Strahle[4] and Chao[5] are consistent with these observations. They concluded that their ability to accurately determine $Q'(x)$ depended on the specific frequency bands being investigated. Although they do not provide a detailed description of the natural modes of their experimental combustor, the frequency band where they encountered the most difficulties and could not recover $Q'(x)$ appears to correspond to the one associated with the fundamental acoustic mode of the combustor. On the other hand, the frequency bands where they found that the investigated technique could be used to determine $Q'(x)$ corresponded to off-resonance frequencies of the combustor. These results are completely consistent with the conclusions of the analysis described earlier.

The good agreement between the conclusions and the previous experimental investigations[2–5] strongly suggests that these conclusions are quite general, i.e., that it is feasible to determine $Q'(x)$ from pressure measurements if the combustion process occurs in a free field or the oscillations are away from any natural acoustic modes of the combustor. On the other hand, $Q'(x)$ cannot be determined from pressure measurements at natural frequencies of the combustor.

Unfortunately, combustion instabilities generally occur when the combustion process excites one or more natural acoustic modes of the combustor. Because it has been shown that it is not possible to recover $Q'(x)$ at these frequencies, it is the conclusion of this study that the investigated technique will not provide a feasible method with which to determine $Q'(x)$ in unstable combustors. It is potentially useful, however, in systems where oscillations are excited at off-resonant frequencies, or for combustion instabilities that do not occur at natural acoustic frequencies of the combustor (such as those due to convected entropy disturbances).

Finally, although the analysis only considered one-dimensional oscillations in the combustion chamber, the results can be generalized to situations involving more complex two- and three-dimensional acoustic oscillations. Because the basic physics of the problem remain unchanged, this method is probably not useful for recovering unsteady heat release processes in this more general case either, for the same reasons as were noted in the one-dimensional case.

Acknowledgments

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